A new class of metrics for spike trains

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Abstract

The distance between a pair of spike trains, quantifying the differences between them, can be measured using various metrics. Here we introduce a new class of spike train metrics, inspired by the Pompeiu-Hausdorff distance. Some of our new metrics (the max-metric and the modulus-metric) are sensitive to differences in the precise timings of spikes in the considered spike trains that cannot be discriminated by existing metrics like the van Rossum distance or the Victor & Purpura distance. The modulus-metric does not depend on any parameters and can be computed using a fast algorithm, in a time that depends linearly on the number of spikes in the two spike trains. We also introduce localized versions of the new metrics, which could have the biologically-relevant interpretation of measuring the differences between spike trains as they are perceived at a particular moment in time by a neuron receiving these spike trains.

1 Introduction

In recent years several spike train distances, some inspired by existing mathematical distances while others not, have been proposed and used to measure the variability of the neural activity (Victor and Purpura, 1996, 1997; van Rossum, 2001; Victor, 2005; Schrauwen et al., 2007; Naud et al., 2011). The distance between two spike trains reflects their similarity or dissimilarity. Because how information is represented in the spatio-temporal patterns of spike times exchanged by neurons is still a heavily debated topic in neuroscience, metrics based on different neural codes are available. Traditionally it was thought that mean firing rate of neurons encapsulated all the relevant information exchanged by neurons. This idea dates back to the work of Adrian (1926) who showed that the firing rate of motor neurons is proportional to the force applied. More recently, scientists have revealed increasing evidence of the importance of precise spike timings to representing information in the brain (Bohte, 2004; VanRullen et al., 2005; Tiesinga et al., 2008). For example, temporally-structured multicell spiking patterns were observed in the hippocampus and cortex, and were associated to memory traces (Nádasdy et al., 1999; Ji and Wilson, 2007) while the coding of information in the phases of spikes

relative to a background oscillation has been observed in many brain regions (Lee et al., 2005; Jacobs et al., 2007; Fries et al., 2007; Montemurro et al., 2008; Siegel et al., 2009). This change in viewpoints from rate codes to spike-time codes is also reflected in spike train metrics.

The most basic metrics are the ones that rely on counting the total number of spikes within a spike train. A major drawback of such an approach is that all the temporal structure is lost. Even though binning techniques were introduced as a way to overcome this loss by dividing the spike train into discrete bins, the temporally–encoded information within a bin was neglected (Geisler et al., 1991). Other, more complex spike train metrics can be obtained by focusing on the precise spike timing instead of their total count. One example is the van Rossum (2001) distance which is calculated by filtering the time series corresponding to the raw spike train with a smoothing kernel and then using the standard Euclidean distance. The choice of kernel is arbitrary and has a high influence on the properties of the metric.

Another metric was introduced by Victor and Purpura (1996, 1997). According to this metric, the distance between two spike trains is given by the minimum cost of basic operations needed to transform one spike train into the other. The basic operations are insertion or deletion of spikes, with a cost of 1, and the shifting of a spike, with a cost of $q|\delta t|$ where q is a parameter and δt the shifting interval. The parameter q clearly influences the behavior of the metric: for $q = 0$ the metric counts the difference in the total number of spikes, while for large values of q the metric returns the number of non-coincident spikes.

Both the van Rossum metric and the Victor & Purpura metric were extended to multi-neuronal cases (Houghton and Sen, 2008; Aronov et al., 2003) to enable the analysis of pattern activity across multiple neurons.

Here we introduce a new class of spike train metrics inspired by the Pompeiu-Hausdorff distance between two non-empty compact sets (Pompeiu, 1905; Hausdorff, 1914). The new spike train metrics yield a result dependent of the exact timing of differences among two spike trains. In the context of the information exchanged by two neurons, each spike may be as important as the spike train itself (Rieke et al., 1997). Therefore, such metrics, based on the specific timing of differences within spike trains, become desirable.

Preliminary results have been presented in abstract form in (Rusu and Florian, 2010).

2 A new class of spike train metrics

We consider bounded, nonempty spike trains of the form

$$T = \{t^{(1)}, \ldots, t^{(n)}\}, \tag{1}$$

where $t^{(i)} \in \mathbb{R}$ are the ordered spike times and $n \in \mathbb{N}^*$ is the number of spikes in the spike train. We consider spike trains with no overlapping spikes, $t^{(i)} \neq t^{(j)}$, $\forall i, j \in \{1, \ldots, n\}$, $i \neq j$. If $n > 1$, then $t^{(i-1)} < t^{(i)}$, $\forall i > 1$. We denote by a and b some bounds of the considered spike trains, i.e. $a \leq t^{(i)} \leq b$, $\forall t^{(i)}$, with $a, b \in \mathbb{R}$, finite, and $a < b$. We denote by $\mathscr{S}_{[a,b]}$ the set of all possible such spike trains bounded by a and b. We study metrics that compute the distances between two spike trains T and $\bar{T}$ from $\mathscr{S}_{[a,b]}$.

The new metrics that we introduce are inspired by the Pompeiu-Hausdorff distance (Pompeiu, 1905; Hausdorff, 1914). When applied to a pair of spike trains, the Pompeiu-Hausdorff distance h returns the largest difference, in absolute value, between the timing of a spike in one train and the timing of the closest spike in the other spike train:

$$h(T,\bar{T}) = \max\left\{\sup_{t\in T}\inf_{\bar{t}\in\bar{T}}|t-\bar{t}|, \sup_{\bar{t}\in\bar{T}}\inf_{t\in T}|t-\bar{t}|\right\}, \tag{2}$$

or, equivalently, the minimal number $\epsilon \geq 0$ such that the closed ϵ-neighborhood of T includes $\bar{T}$ and the closed ϵ-neighborhood of $\bar{T}$ includes T:

$$h(T,\bar{T}) = \inf\left\{\epsilon \text{ such that } |t-\bar{t}| \leq \epsilon,\ \forall t \in T,\ \forall \bar{t} \in \bar{T}\right\}. \tag{3}$$

Another equivalent form of the Pompeiu-Hausdorff distance is the following (Papadopoulos, 2005, pp. 105–110; Rockafellar and Wets, 2009, pp. 117–118; Deza and Deza, 2009, pp. 47–48):

$$h(T,\bar{T}) = \sup_{x\in\mathbb{R}}\left|\inf_{t\in T}|t-x| - \inf_{\bar{t}\in\bar{T}}|\bar{t}-x|\right|. \tag{4}$$

We introduce a distance d between an arbitrary timing $x \in \mathbb{R}$ and a spike train T:

$$d(x,T) = \inf_{t\in T}|t-x|. \tag{5}$$

Eq. 2 can then be rewritten as

$$h(T,\bar{T}) = \max\left\{\sup_{t\in T} d(t,\bar{T}), \sup_{\bar{t}\in\bar{T}} d(\bar{t},T)\right\} \tag{6}$$

and Eq. 4 as

$$h(T,\bar{T}) = \sup_{x\in\mathbb{R}}\left|d(x,T) - d(x,\bar{T})\right|. \tag{7}$$

We also have (Appendix A):

$$h(T,\bar{T}) = \sup_{x\in[a,b]}\left|d(x,T) - d(x,\bar{T})\right|. \tag{8}$$

The Pompeiu-Hausdorff metric has a quite poor discriminating power, as for many variations of the spike trains the distances will be equal and any spike train space endowed with this metric would be highly clusterized. Our new metrics generalize the form of the Pompeiu-Hausdorff distance given in Eq. 8, by introducing features that are more sensitive to spike timings.

2.1 The max-metric

We consider $\mathbb{B}$ to be the space of all continuous, positive functions $\mathcal{H}\colon [0, b-a] \to \mathbb{R}^+$ that satisfy the condition, $\forall u \in [a,b]$,

$$\int_a^b \mathcal{H}(|u-s|)\,\mathrm{d}s > 0. \tag{9}$$

Because

$$\int_a^b \mathscr{H}(|u-s|)\,\mathrm{d}s = \int_a^u \mathscr{H}(u-s)\,\mathrm{d}s + \int_u^b \mathscr{H}(s-u)\,\mathrm{d}s \tag{10}$$

$$= \int_0^{u-a} \mathscr{H}(s)\,\mathrm{d}s + \int_0^{b-u} \mathscr{H}(s)\,\mathrm{d}s, \tag{11}$$

and because $\mathscr{H}$ is continuous, a sufficient condition for satisfying Eq. 9 is that $\mathscr{H}(0) > 0$. On compact sets, continuous functions are bounded (Protter, 1998, p. 56). We denote by m the upper bound of $\mathscr{H}$ on the interval $[0, b-a]$, i.e.

$$0 \le \mathscr{H}(x) < m < \infty,\ \forall\ x \in [0, b-a]. \tag{12}$$

In typical applications, $\mathscr{H}(x)$ has a maximum for $x = 0$ and is a decreasing function of x, for example an exponential,

$$\mathscr{H}_E(x) = \frac{1}{\tau}\,\exp\left(-\frac{x}{\tau}\right), \tag{13}$$

or a Gaussian,

$$\mathscr{H}_G(x) = \frac{1}{\tau\,\sqrt{2\,\pi}}\,\exp\left(-\frac{x^2}{2\,\tau^2}\right), \tag{14}$$

with τ a positive parameter.

We introduce the max-metric as

$$d_m(T,\bar{T}) = \int_a^b \sup_{x\in[a,b]} \left\{|d(x,T) - d(x,\bar{T})|\,\mathscr{H}(|s-x|)\right\} \mathrm{d}s. \tag{15}$$

The max-metric integrates, through the variation of s along the interval $[a,b]$ that contains the two spike trains, the maximum of the difference, in absolute value, between the distances from a point x in that interval to the two spike trains, weighted by the kernel $\mathscr{H}(|s-x|)$ which focuses locally around s. Figure 1 shows how the distance between two spike trains is computed using the max-metric.

The max-metric is a generalization of the Pompeiu-Hausdorff distance, since in the particular case that $\mathscr{H}(\cdot) = 1/(b-a)$ we have $d_m(T,\bar{T}) = h(T,\bar{T})$.

In Appendix B we show that d_m is finite and that it satisfies the properties of a metric. We also show that regardless of the kernel $\mathscr{H}$ all the max-metrics are topologically equivalent to each other (O'Searcoid, 2007, p. 229) because they are equivalent to the Pompeiu-Hausdorff distance. Each metric will generate the same topology and thus any topological property is invariant under an homeomorphism. This means that the metrics generate the same convergent sequences in the space of spike trains $\mathscr{S}_{[a,b]}$. Thus, learning rules derived from these metrics will converge in the same way, regardless of the choice of $\mathscr{H}$.

2.2 The modulus-metric

We define the modulus-metric as

$$d_o(T,\bar{T}) = \int_a^b |d(s,T) - d(s,\bar{T})|\,\mathrm{d}s. \tag{16}$$

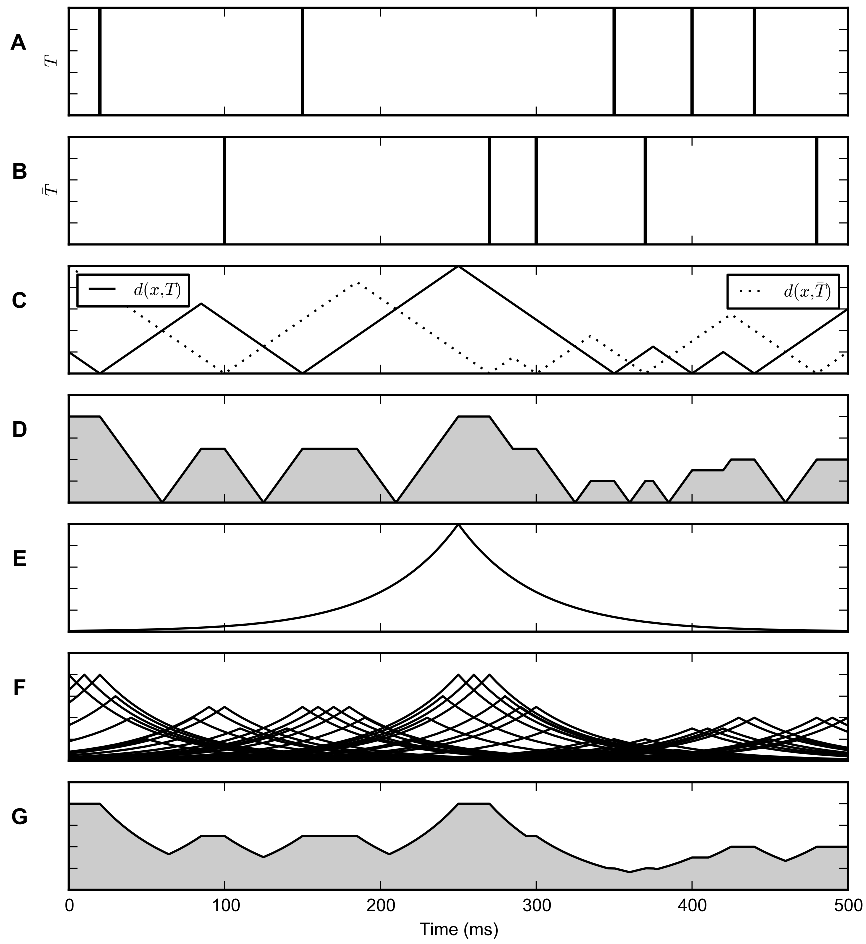

Figure 1: The modulus-metric and the max-metric. (A) Spike train $T = \{20, 150, 350, 400, 440\}$ ms. Each spike time is represented as a vertical bar. (B) Spike train $\bar{T} = \{100, 270, 300, 370, 480\}$ ms. (C) The distance between a timing x and the spike trains $d(x,T)$ and $d(x,\bar{T})$, as a function of x. (D) The difference $\left|d(x,T) - d(x,\bar{T})\right|$ as a function of x. The modulus-metric distance $d_o(T, \bar{T})$ is the area under this curve. (E) The kernel $\mathscr{H}(|s-x|)$ as a function of s with a fixed $x = 250$ ms. $\mathscr{H}$ is an exponential (Eq. 13) with a decay constant $\tau = 50$ ms. (F) The weighted difference $\left|d(x,T) - d(x,\bar{T})\right| \mathscr{H}(|s-x|)$ as function of s for discrete values of x. (G) The supremum of the weighted difference, $\sup_{x \in [a,b]} \left\{\left|d(x,T) - d(x,\bar{T})\right| \mathscr{H}(|s-x|)\right\}$, as a function of s. The max-metric distance $d_m(T, \bar{T})$ is the area under this curve.

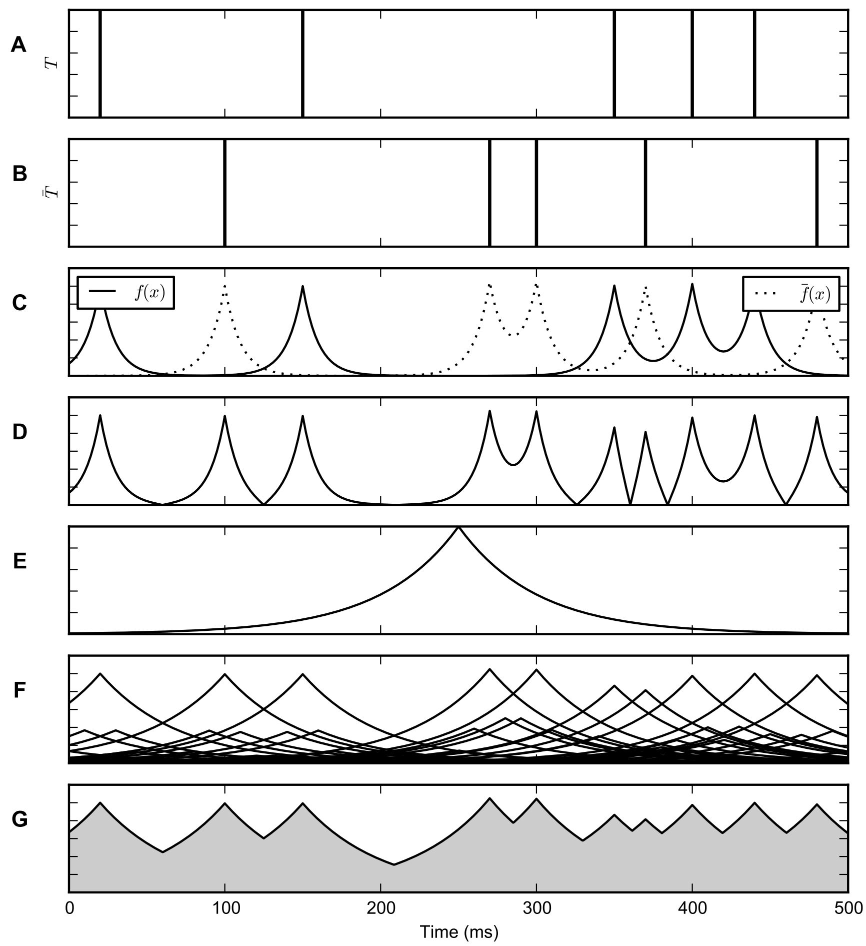

Figure 2: The convolution max-metric. (A) Spike train $T = \{20, 150, 350, 400, 440\}$ ms. Each spike time is represented as a vertical bar. (B) Spike train $\bar{T} = \{100, 270, 300, 370, 480\}$ ms. (C) The spike trains T and $\bar{T}$ filtered with an exponential kernel (Eq. 19) with a decay constant $\tau = 10$ ms. (D) The difference $\left|f(x) - \bar{f}(x)\right|$ as a function of x. (E) The kernel $\mathscr{H}(|s-x|)$ as a function of s with a fixed $x = 250$ ms. $\mathscr{H}$ is an exponential (Eq. 13) with a decay constant $\tau = 50$ ms. (F) The weighted difference $\left|f(x) - \bar{f}(x)\right| \mathscr{H}(|s-x|)$ as function of s for discrete values of x. (G) The supremum of the weighted difference, $\sup_{x\in[a,b]} \left\{\left|f(x) - \bar{f}(x)\right| \mathscr{H}(|s-x|)\right\}$. The convolution max-metric distance $d_c(T, \bar{T})$ is the area under this curve.

Figure 1 A–D shows how the distance between two spike trains is computed using the modulus-metric. The modulus-metric is a particular case of the max-metric in the limit that $\mathscr{H}$ is

$$\mathscr{H}(x) = \begin{cases} 1, & \text{if} \quad x = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

The modulus-metric does not depend on any kernels or parameters and it also allows a fast computer implementation in linear complexity. This is because the graph of the function $\phi(s) = |d(s,T) - d(s,\bar{T})|$ is made out of line segments that join or end in the following points: all timings of spikes in the two spike trains T and $\bar{T}$; the time moments that lie at the middle of the interval between two neighboring spikes from the same spike train; the time moments that lie at the middle of the interval between a pair of neighboring spikes where the two spikes belong to different spike trains; and the bounds a and b. This is exemplified in Figures 1 D and 3 D. We denote by $\mathscr{P}$ the set of these points. In order to compute the integral of this function $d_o = \int_a^b \phi(s)\,\mathrm{d}s$, it is sufficient to compute the function at the points from $\mathscr{P}$. Since between these points the function is linear, the integral can be then computed exactly.

Algorithm 1 presents a quick-and-dirty implementation of the d_o metric in pseudocode. In this algorithm, the function ϕ is computed in a set of points that includes $\mathscr{P}$ but also other points.

In the optimized Algorithm 2, the set $\mathscr{P}$ as well as the value of ϕ in the points of $\mathscr{P}$ is computed with a single pass through the spikes in the two spike trains. The algorithm's duration depends linearly on the number of spikes in the two spike trains, $n + \bar{n}$. A Python implementation of this optimized algorithm is freely available at https://github.com/modulus-metric/.

It can be shown that the distance d_o is finite and that it satisfies the properties of a metric by particularizing the proofs in Appendix E with $\mathscr{L}(x) = 1$, $\forall x \in \mathbb{R}$.

2.3 The convolution max-metric

The max-metric can also be given in a convolution form. To construct this form of the metric we consider an arbitrary continuous, smooth, positive kernel $\mathscr{K} : \mathbb{R} \to \mathbb{R}^+$, with the properties that it is strictly increasing for $x < 0$ and strictly decreasing for $x > 0$ and that

$$0 \le \mathscr{K}(x) \le p \text{ for every } x \in \mathbb{R}, \text{ and } \mathscr{K}(0) > 0, \tag{18}$$

where p is a positive, finite bound. Typically, $\mathscr{K}$ is an exponential,

$$\mathscr{K}_E(x) = \exp\left(-\frac{|x|}{\tau}\right), \tag{19}$$

with τ being a positive parameter. We convolve the two spike trains T and $\bar{T}$ with the filtering kernel $\mathscr{K}$ to obtain

$$f(x) = \sum_{i=1}^{n} \mathscr{K}(x - t^{(i)}) \tag{20}$$

$$\bar{f}(x) = \sum_{i=1}^{\bar{n}} \mathscr{K}(x - \bar{t}^{(i)}). \tag{21}$$

Input: The pair of nonempty spike trains T_1, T_2 and the bounds a and b.
Output: The distance $d_o(T_1, T_2)$ between the spike trains.

```
⟨T1, T2 and P are ordered sets of real numbers, indexed starting from 0.⟩
n1 := length(T1); n2 := length(T2);
P := sort(T1 ∪ T2);
M := ∅;
for i := 1...n1 − 1 do
    M := M ∪ {(T1[i] + T1[i − 1])/2};
for i := 1...n2 − 1 do
    M := M ∪ {(T2[i] + T2[i − 1])/2};
for i := 1...length(P) − 1 do
    M := M ∪ {(P[i] + P[i − 1])/2};
P := sort(P ∪ M ∪ {a, b});
d_o := 0; i1 := 0; i2 := 0;
⟨s is the currently considered point from P. φ is the value at s of the integrated
function |d(s, T1) − d(s, T2)|. s_p is the previously considered point from P, and φ_p is
the value at s_p of the integrated function. i1 is the index of the first spike in T1
having a timing that is greater than s, if there is such a spike, or the index of the last
spike of T1, otherwise. i2 is computed analogously for T2.⟩
s_p := a; φ_p := |T1[0] − T2[0]|;
for i := 1...length(P) − 1 do
    s := P[i];
    while s ≥ T1[i1] and i1 < n1 − 1 do
        i1 := i1 + 1;
    while s ≥ T2[i2] and i2 < n2 − 1 do
        i2 := i2 + 1;
    d1 := b − a; d2 := b − a;
    if i1 > 0 then
        d1 := s − T1[i1 − 1];
    d1' = |T1[i1] − s|;
    if d1' < d1 then
        d1 := d1';
    if i2 > 0 then
        d2 := s − T2[i2 − 1];
    d2' = |T2[i2] − s|;
    if d2' < d2 then
        d2 := d2';
    ⟨We now have d1 = d(s, T1) and d2 = d(s, T2) and we can compute the value of φ
    at s:⟩
    φ := |d1 − d2|;
    ⟨The integration is performed here:⟩
    d_o := d_o + (s − s_p)(φ + φ_p)/2;
    s_p := s; φ_p := φ;
return d_o;
```

Algorithm 1: A quick-and-dirty algorithm for computing the modulus-metric distance d_o between two spike trains T_1 and T_2. The text surrounded by $\langle \ldots \rangle$ represents comments.

Input: The pair of non-empty spike trains T_1, T_2 and the bounds a and b.
Output: The distance $d_o(T_1, T_2)$ between the spike trains.
⟨T_1 and T_2 are ordered sets of real numbers, indexed starting from 0. i_1 and i_2 are the indices of the currently processed spikes in the two spike trains. p_1 and p_2 are the indices of the previously processed spikes in the two spike trains. p is the index of the spike train to which the previously processed spike belonged (1 or 2), after at least one spike has been processed, or 0 otherwise.⟩
$i_1 := 0; i_2 := 0; p_1 := 0; p_2 := 0; p := 0;$
$n_1 := \texttt{length}(T_1);\ n_2 := \texttt{length}(T_2);$
⟨P is an array of structures (s, ϕ) consisting of a ordered pair of numbers.⟩
$P := \{(s \mapsto a,\ \phi \mapsto |T_1[0] - T_2[0]|)\};$
⟨Process the spikes until the end of one of the spike trains is reached.⟩
while $i_1 < n_1$ `and` $i_2 < n_2$ **do**
  if $T_1[i_1] \le T_2[i_2]$ **then**
    proc1(1, 2);
  else
    proc1(2, 1);
⟨Process the rest of the spikes in the spike train that has not been fully processed.⟩
while $i_1 < n_1$ **do**
  proc2(1, 2);
while $i_2 < n_2$ **do**
  proc2(2, 1);
$P := P \bigcup \{(s \mapsto b,\ \phi \mapsto |T_1[n_1 - 1] - T_2[n_2 - 1]|)\};$
⟨Sort P. Elements of P are sorted according to their value of s.⟩
$P := \mathrm{sort}(P);$
⟨Perform the integration.⟩
$d_o := 0;$
for $i := 1 \ldots \texttt{length}(P) - 1$ **do**
  $d_o := d_o + (P[i].s - P[i-1].s)(P[i].\phi + P[i-1].\phi)/2;$
return d_o;

Algorithm 2: An optimized algorithm for computing the modulus-metric distance d_o between two spike trains T_1 and T_2. The text surrounded by ⟨…⟩ represents comments.

Input: The indices j and k of the two sorted spike trains T_j, T_k; $j,k \in \{1,2\}$, $j \neq k$.
Data: Uses as global variables: the indices i_j, i_k of the current spikes in the two spike trains; the indices p_j, p_k of the previously processed spikes in the two spike trains; the index p of the spike train to which the previously processed spike belonged (1 or 2; if no spike has been previously processed, $p = 0$); the data structure P. We should have here $T_j[i_j] \leq T_k[i_k]$. If $p \neq 0$, we should have $T_p[i_p] \leq T_j[i_j]$.
Result: Performs part of the processing needed for creating P. The procedure is used when processing has not reached the end of one of the spike trains.

if $i_j > 0$ **then**

  ⟨Adds to P the timing situated at the middle of the interval between the currently processed spike and the previous spike in the same spike train.⟩
  $t := (T_j[i_j] + T_j[i_j - 1])/2$;
  ⟨We have $d(t, T_j) = T_j[i_j] - t = t - T_j[i_j - 1] = (T_j[i_j] - T_j[i_j - 1])/2$.⟩
  $P := P \cup \left\{\left(s \mapsto t,\ \phi \mapsto \left|(T_j[i_j] - T_j[i_j - 1])/2) - d(t, k, i_k)\right|\right)\right\}$;

if $p = k$ **then**

  ⟨If the previously processed spike was one from the other spike train than the spike currently processed, adds to P the timing situated at the middle of the interval between the currently processed spike and the previously processed spike.⟩
  $t := (T_j[i_j] + T_k[p_k])/2$;
  ⟨Since t is at equal distance to the closest spikes in the two spike trains, $T_j[i_j]$ and $T_k[p_k]$, we have $d(t, T_j) = d(t, T_k)$ and $\phi(t) = 0$.⟩
  $P := P \cup \left\{\left(s \mapsto t,\ \phi \mapsto 0\right)\right\}$;

⟨Adds to P the currently processed spike.⟩
$t := T_j[i_j]$;
⟨We have $d(t, T_j) = 0$. If at least one spike from T_k has been processed, we have $T_k[p_k] \leq t \leq T_k[i_k]$, with $i_k = p_k + 1$, and thus $d(t, T_k) = \min(|t - T_k[p_k]|,\ T_k[i_k] - t)$. If no spike from T_k has been processed, we have $p_k = i_k = 0$, and the previous formula for $d(t, T_k)$ still holds.⟩
$P := P \cup \left\{\left(s \mapsto t,\ \phi \mapsto \min(|t - T_k[p_k]|,\ T_k[i_k] - t)\right)\right\}$;
$p_j := i_j$;
$i_j := i_j + 1$;
$p := j$;

Procedure proc1(j, k).

Input: The indices j and k of the two sorted spike trains T_j, T_k; $j, k \in \{1, 2\}$, $j \neq k$.

Data: Uses as global variables: the index i_j of the current spike in T_j; the index p_k of the previously processed spike in T_k; the index p of the spike train to which the previously processed spike belonged (1 or 2); the data structure P. Here, p_k should be the index of the last spike in spike train T_k. We should have $T_k[p_k] \leq T_j[i_j]$.

Result: Performs part of the processing needed for creating P. The procedure is used when processing has reached the end of spike train T_k.

if $i_j > 0$ **then**

  ⟨Adds to P the timing situated at the middle of the interval between the currently processed spike and the previous spike in the same spike train.⟩

  $t := (T_j[i_j] + T_j[i_j - 1])/2$;

  ⟨We have $d(t, T_j) = T_j[i_j] - t = t - T_j[i_j - 1] = (T_j[i_j] - T_j[i_j - 1])/2$.⟩

  $P := P \cup \left\{\left(s \mapsto t,\, \phi \mapsto \left|(T_j[i_j] - T_j[i_j - 1])/2) - d(t, k, p_k)\right|\right)\right\}$;

if $p = k$ **then**

  ⟨If the previously processed spike was one from the other spike train than the spike currently processed (i.e., the last spike in the spike train that has been fully processed), adds to P the timing situated at the middle of the interval between the currently processed spike and the previously processed spike.⟩

  $t := (T_j[i_j] + T_k[p_k])/2$;

  ⟨Since t is at equal distance to the closest spikes in the two spike trains, $T_j[i_j]$ and $T_k[p_k]$, we have $d(t, T_j) = d(t, T_k)$ and $\phi(t) = 0$.⟩

  $P := P \cup \left\{\left(s \mapsto t,\, \phi \mapsto 0\right)\right\}$;

⟨ Adds to P the currently processed spike. ⟩

$t := T_j[i_j]$;

⟨ We have $d(t, T_j) = 0$. We have $T_k[p_k] \leq t$ and the spike at p_k is the last one in T_k, and thus $d(t, T_k) = t - T_k[p_k]$.⟩

$P := P \cup \left\{\left(s \mapsto t,\, \phi \mapsto t - T_k[p_k]\right)\right\}$;

$i_j := i_j + 1$;

$p := j$;

Procedure proc2(j, k).

Input: A timing t, the index $k \in \{1,2\}$ of a sorted spike train T_k, and the index i of a spike in T_k, such that either $t \le T_k[i]$ or i is the index of the last spike of T_k.
Output: The distance $d(t, T_k)$ between the timing t and the spike train T_k.

```
d := |T_k[i] − t|;
j := i − 1;
while j ≥ 0 and |T_k[j] − t| ≤ d do
    d := |T_k[j] − t|;
    j := j − 1;
return d;
```

Function $\mathrm{d}(t, k, i)$.

We denote by $\mathscr{F}_{[a,b]}$ the set of all possible filtered spike trains from $\mathscr{S}_{[a,b]}$.

We also consider a function $\mathscr{H} \in \mathbb{B}$ that is strictly positive, that is derivable on $(0, b-a)$ and that has a bounded derivative.

The convolution max-metric is defined as

$$d_c(T, \bar{T}) = \int_a^b \sup_{x \in [a,b]} \left\{ |f(x) - \bar{f}(x)| \mathscr{H}(|s-x|) \right\} \mathrm{d}s. \tag{22}$$

Figure 2 shows how the distance d_c between two spike trains is computed. In Appendix C we show that d_c is finite and that it satisfies the properties of a metric.

3 Localized metrics

In the case of the max metric, with or without convolution, the use of the kernel $\mathscr{H}$ served the purpose of providing a local perspective, around each point within $[a, b]$, of the distance between the spike trains. These local perspectives were then integrated in the final distance. In this section we introduce different metrics that also depend on a kernel $\mathscr{L}$, but for which the kernel has a different purpose. More precisely, the kernel may be regarded as a magnifying glass to be used to focus on one specific area of the spike trains. The kernel should be a continuous, positive function, $\mathscr{L} : [0, b-a] \to \mathbb{R}^+$, such that

$$\mathscr{L}(x) > 0, \ \forall x \in (0, b-a]. \tag{23}$$

As $\mathscr{H}$, because $\mathscr{L}$ is a continuous function with bounded support, it is bounded, i.e.

$$0 \le \mathscr{L}(x) < m < \infty, \ \forall \ x \in [0, b-a]. \tag{24}$$

Such a metric is biologically relevant if, for example, we take into consideration how a neuron responds to input spikes. Recent spikes influence more the neuron than old ones. If we would like to measure the distance between two spike trains according to how the differences between them influence the activity of a neuron at a particular moment of time, recent differences should be taken into account with a greater weight than differences in the distant past. For the localized metrics, $\mathscr{L}$ could thus model the shape of postsynaptic potentials (PSP) that reflects the dynamics of the effect of one presynaptic spike on the studied neuron. Thus, $\mathscr{L}$ could typically be an exponential, $\mathscr{L}_E = \mathscr{H}_E$ (Eq. 13), an alpha function,

$$\mathscr{L}_\alpha(x) = \frac{x}{\tau^2} \exp\left(-\frac{x}{\tau}\right), \tag{25}$$

a difference between two exponentials,

$$\mathscr{L}_D(x) = \frac{\tau}{\tau - \tau_s} \left[\exp\left(-\frac{x}{\tau}\right) - \exp\left(-\frac{x}{\tau_s}\right)\right], \tag{26}$$

or, if we model the postsynaptic potential generated in an integrate-and-fire neuron by a synaptic current that is a difference between two exponentials,

$$\begin{aligned}\mathscr{L}_I(x) = \frac{\tau}{\tau_s - \tau_r} &\left\{\frac{\tau_s}{\tau - \tau_s}\left[\exp\left(-\frac{x}{\tau}\right) - \exp\left(-\frac{x}{\tau_s}\right)\right]\right. \\ &- \left.\frac{\tau_r}{\tau - \tau_r}\left[\exp\left(-\frac{x}{\tau}\right) - \exp\left(-\frac{x}{\tau_r}\right)\right]\right\},\end{aligned} \tag{27}$$

where τ, τ_s, and τ_r are positive parameters.

3.1 Localized max-metric

We introduce the localized max-metric as

$$d_l(T, \bar{T}) = \int_a^b \mathscr{L}(b-s) \sup_{x \in [s,b]} |d(x,T) - d(x,\bar{T})| \,\mathrm{d}s. \tag{28}$$

Figure 3 shows how the distance d_l between two spike trains is computed. The differences between the spike trains that account the most for the distance are those that are close to b. The shape of $\mathscr{L}$ has a high impact on the behavior of the metric.

In Appendix D we show that the distance d_l is finite and that it satisfies the properties of a metric.

3.2 Localized modulus-metric

The modulus metric can also be given in a localized form:

$$d_n(T, \bar{T}) = \int_a^b |d(s,T) - d(s,\bar{T})| \,\mathscr{L}(b-s) \,\mathrm{d}s. \tag{29}$$

In Appendix E we show that d_n is finite and that it satisfies the properties of a metric.

3.3 Localizing the van Rossum metric

A localization by a kernel $\mathscr{L}$ similar to the one we applied in Eqs. 28 and 29 can also be applied to existing metrics. Let $T, \bar{T} \in \mathscr{S}_{[a,b]}$. Consider the van Rossum (2001) distance defined as

$$d_R(T, \bar{T}) = \int_{-\infty}^{\infty} (g(s) - \bar{g}(s))^2 \,\mathrm{d}s, \tag{30}$$

where

$$g(s) = \sum_{i=1}^{n} H(s - t^{(i)}) \,\mathscr{K}_E(s - t^{(i)}) \tag{31}$$

$$\bar{g}(s) = \sum_{i=1}^{\bar{n}} H(s - \bar{t}^{(i)}) \,\mathscr{K}_E(s - \bar{t}^{(i)}), \tag{32}$$

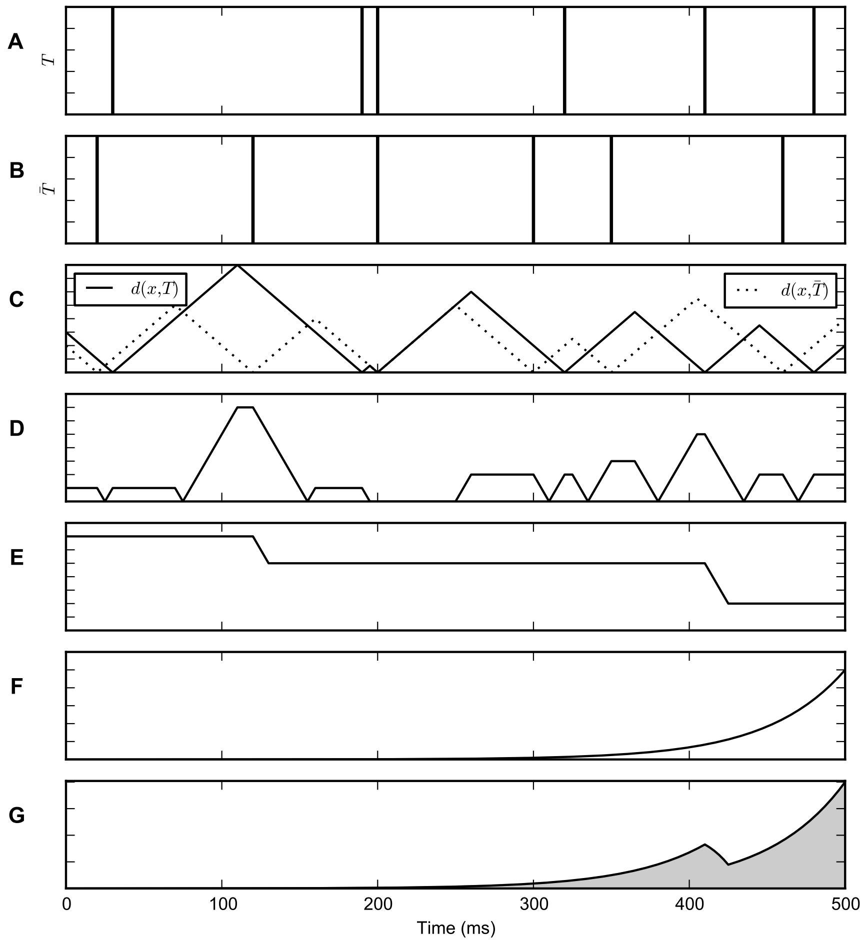

Figure 3: The localized max-metric. (A) Spike train $T = \{30, 190, 200, 320, 410, 480\}$ ms. Each spike time is represented as a vertical bar. (B) Spike train $\bar{T} = \{20, 120, 200, 300, 350, 460\}$ ms. (C) The distance between a timing x and the spike trains $d(x, T)$ and $d(x, \bar{T})$ as function of x. (D) The difference $\left|d(x, T) - d(x, \bar{T})\right|$ as a function of x. (E) The supremum, $\sup_{x\in[s,b]} \left|d(x, T) - d(x, \bar{T})\right|$. (F) The kernel $\mathscr{L}(b - s)$ as function of s, an exponential (Eq. 13) with a decay constant $\tau = 50$ ms. (G) The supremum weighted by the kernel, $\mathscr{L}(b - s) \sup_{x\in[s,b]} |d(x, T) - d(x, \bar{T})|$, as a function of s. The localized max-metric distance $d_l(T, \bar{T})$ is the area under this curve.

H is the Heaviside step function, $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$, and $\mathscr{K}_E$ is defined in Eq. 19. When localized with $\mathscr{L}$ the distance becomes

$$d_{Rl}(T, \bar{T}) = \int_{-\infty}^{b} (g(s) - \bar{g}(s))^2 \mathscr{L}(b - s)\,\mathrm{d}s. \tag{33}$$

Here $\mathscr{L}$ may be chosen to have the same qualitative properties as the kernels used in Eqs. 25–27.

4 Simulation results

We analyzed the behavior of the introduced metrics through computer simulations using simple setups. Across all simulations, $\mathscr{H}$ was an exponential (Eq. 13) with $\tau = 10$ ms. For the localized metrics d_l and d_n, $\mathscr{L}$ also was an exponential (Eq. 13) with $\tau = 20$ ms. The convolution kernels for the d_c and van Rossum distances were chosen as exponentials (Eq. 19) with $\tau = 10$ ms. For the Victor & Purpura distance we set $q = 0.1$ ms^{-1}. We set $a = 0$ ms and b the maximum length of spike trains (either 200 or 500 ms).

First, we computed the distances between a particular spike train T and a spike train $\bar{T}$ obtained from it by either inserting or shifting one spike. In the insertion case, $\bar{T}$ was generated by inserting a spike into T at various timings. In the shifting case, $\bar{T}$ was generated by shifting a particular spike of T. The distance was plotted against the time of the inserted spike or of the shifted spike to see how the change is reflected by the metrics. To compute the distance we used the introduced metrics, a simple spike count distance (c), the Pompeiu-Hausdorff distance (h) and the Victor and Purpura (1996, 1997) (d_{VP}) and van Rossum (2001) (d_R) distances. We used $b = 200$ ms. The spike count distance is defined as

$$c(T, \bar{T}) = \frac{|n - \bar{n}|}{\max(n, \bar{n})}, \tag{34}$$

where n and $\bar{n}$ are the number of spikes in each train.

The results for the insertion case are presented in Figure 4. The Victor & Purpura distance was constant since the cost of adding and removing a spike is fixed at 1 regardless of its timing. Similarly, the van Rossum metric was insensitive to the time of the inserted spike, a result which can be also shown analytically (van Rossum, 2001). The spike count distance remained constant regardless of where the spike was inserted. The results were qualitatively different in the case of our newly introduced distances, with the exception of the convolution max-metric, showing how they depend on the spike time differences across spike trains. In the case of the Pompeiu-Hausdorff distance, max-metric, modulus-metric, and of the localized variants of the max-metric and the modulus-metric, the insertion time of the spike had a significant impact on the outcome (Figure 4). It can also be seen that the localized distances were strongly influenced by the shape of the $\mathscr{L}$ kernel.

The results for the shifting case are presented in Figure 5. When the spike at $t^{(4)}$ was shifted, the Victor & Purpura and van Rossum distances were dependent only on the width of the shifting interval. These results are confirmed by analytical derivations (Victor and Purpura, 1996; van Rossum, 2001). As in the previous case, the spike count distance was insensitive to the shift operation and remained zero since the number of spikes did not change. In contrast, our newly introduced distances, with the exception of the convolution max-metric, showed a dependence

not only on the width of the shifting interval but also on the particular timing of the shifted spike.

Second, we explored the correlation between the newly introduced metrics and the classical Victor & Purpura and van Rossum distances. We generated a 500 ms Poisson spike train with a firing rate of 20 Hz. From this spike train, we generated a new one by adding a Gaussian jitter with zero mean and 20 ms. We considered only generated and jittered spike trains that contained 10 spikes. We then measured the distance between the original and the jittered spike train using various metrics. We repeated this for 5,000 samples, where for each sample the original spike train and the jitter were generated randomly. The results are displayed in Figure 6. Table 1 shows the correlation coefficients between the max-metric, the modulus-metric, the convolution max-metric and, respectively, the van Rossum and Victor & Purpura distances.

Table 1: Correlation coefficients between the introduced distances and, respectively, the van Rossum and Victor & Purpura distances, computed from data presented in Figure 6.

Distance	d_R correlation coefficient	d_{VP} correlation coefficient
d_m	0.5391	0.6677
d_o	0.5405	0.6512
d_c	0.8450	0.9535
d_R	1.0000	0.8475
d_{VP}	0.8475	1.0000

5 Discussion

The max-metric and the modulus-metric can discriminate differences in the precise timings of individual spikes that are not discriminated by the classical van Rossum and Victor & Purpura distances (Figures 4, 5). The max-metric and the modulus-metric have qualitatively similar properties. While the max-metric depends on a kernel $\mathscr{H}$ which can be particularized to cause distinct behaviors, and the van Rossum and Victor & Purpura distances also depend on parameters that must be chosen by their users, the modulus-metric does not depend on any parameters. We have also shown that the modulus-metric can be computed through a fast algorithm that operates in a time that depends linearly on the number of spikes in the considered spike trains.

The convolution-metric that we introduced, although analytically similar to the max-metric, is qualitatively similar to the van Rossum distance. A qualitative difference between the convolution-metric and the van Rossum distance appears when the differences between the spike trains are localized near the ends of the integration interval, and this is a simple consequence of the difference between the bounded integration interval for the convolution-metric and the infinite integration interval for the van Rossum distance.

We have considered only spike trains having non-overlapping spikes. If we relax this constraint, for our newly introduced distances, with the exception of the convolution max-metric, we get a zero distance between a spike train and a second one generated from the first by adding an extra spike to the first, overlapping

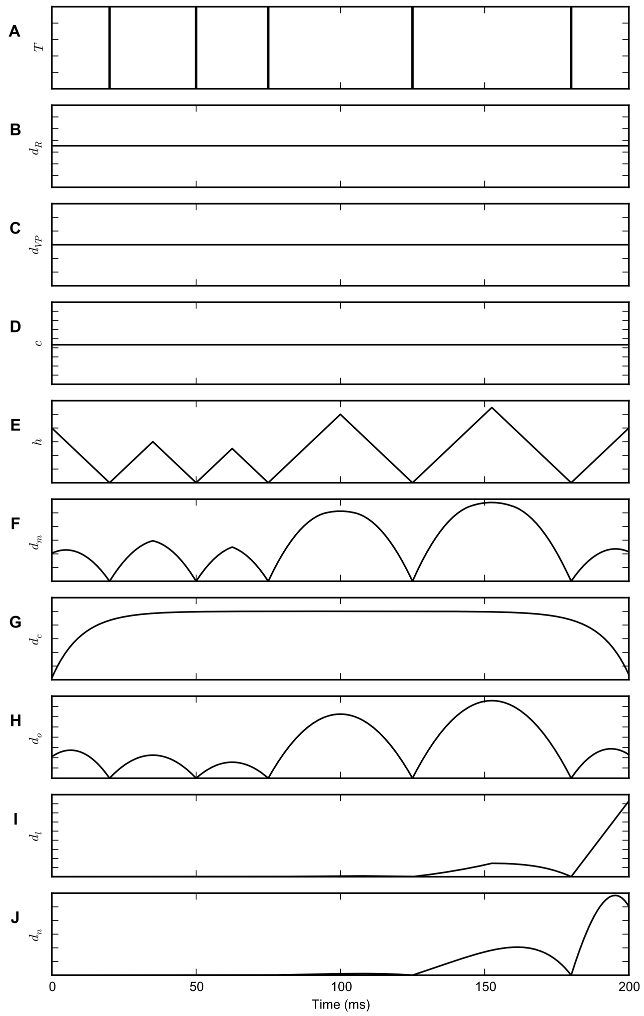

Figure 4: Metric comparison. We computed the distance between the spike train $T = \{20, 50, 75, 125, 180\}$ ms and one obtained from this spike train by inserting a spike at different locations. At each time $s = 0 \ldots 200$ ms a spike was inserted to generate $\bar{T}$ and the distance between T and $\bar{T}$ was computed and plotted against s. (A) The spike train T. (B) The van Rossum distance. (C) The Victor & Purpura distance. (D) The spike count distance. (E) The Pompeiu-Hausdorff distance. (F) The max-metric. (G) The convolution max-metric. (H) The modulus-metric. (I) The localized max-metric. (J) The localized modulus-metric.

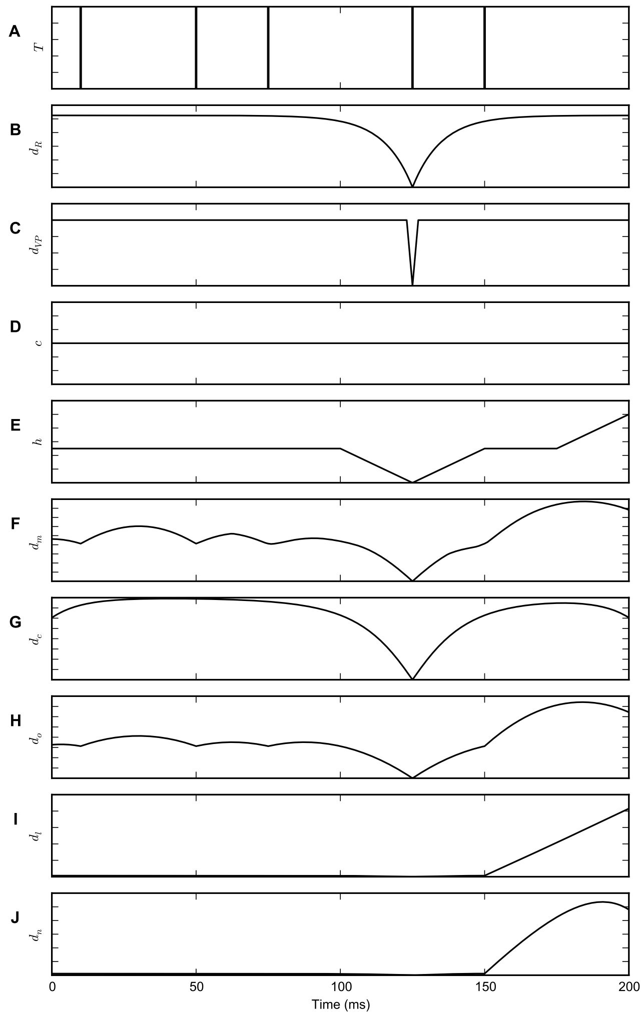

Figure 5: Metric comparison. We computed the distance between the spike train $T = \{10, 50, 75, 125, 150\}$ ms and one obtained from this spike train by shifting the spike at $t^{(4)} = 125$ ms. The spike was shifted at timings $s = 0 \dots 200$ ms to generate $\bar{T}$ and the distance between T and $\bar{T}$ was computed and plotted against s. (A) The spike train T. (B) The van Rossum distance. (C) The Victor & Purpura distance. (D) The spike count distance. (E) The Pompeiu-Hausdorff distance. (F) The max-metric. (G) The convolution max-metric. (H) The modulus-metric. (I) The localized max-metric. (J) The localized modulus-metric.

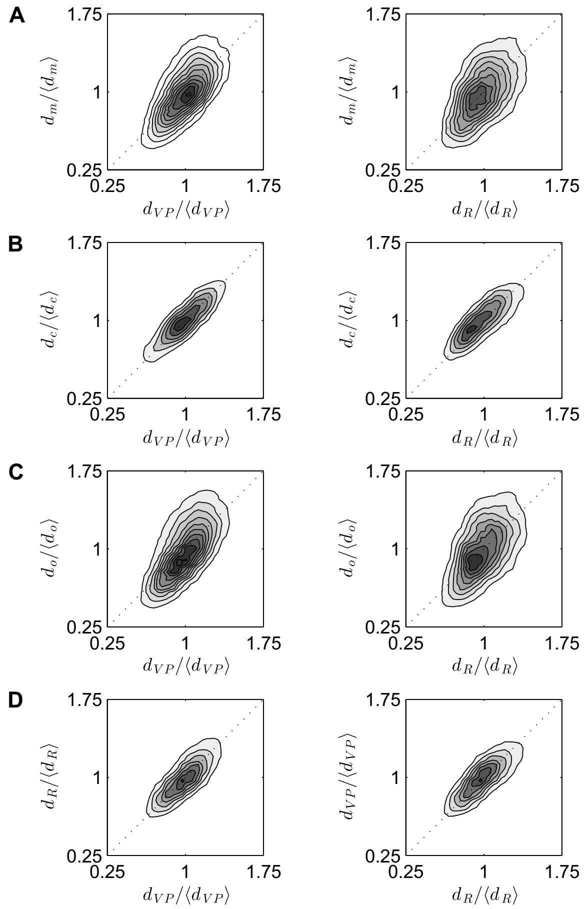

Figure 6: Correlations between the distances computed with our newly introduced metrics and, respectively, the Victor & Purpura (left column) and the van Rossum (right column) metrics. A 20 Hz Poisson spike train was generated over 500 ms and then jitter drawn from a Gaussian distribution with zero average and 20 ms variance was applied to it. The distances between the original and jittered spike train were then computed. The process was repeated 5,000 times. All distances were normalized to the mean value across samples. (A) Max-metric. (B) Convolution max-metric. (C) Modulus-metric. (D) Victor & Purpura / van Rossum.

an existing spike (Figure 4). This is due to the distance d between an arbitrary timing and a spike train in Eq. 5 which does not distinguish between overlapping spikes in a train. Thus, if we relax the constraint of not allowing non-overlapping spikes, these distances became pseudo-metrics because we can have zero distance between two spike trains that differ through overlapping spikes. However, the case of overlapping spikes is biologically implausible if we consider spike trains fired by single neurons. If it is enough that the distances are pseudo-metrics, we may also relax some of the conditions of the kernels, such as the requirement for $\mathscr{L}$ to be strictly positive on $(0, b-a]$ or the conditions on $\mathscr{K}$.

For our metrics, when the integrating interval $[a, b]$ extends beyond the interval covered by extremes of the spike trains, e.g. $[\min(t^{(1)}, \bar{t}^{(1)}), \max(t^{(n)}, \bar{t}^{(\bar{n})})]$ for a pair of spike trains, the result of the integration in the area not covered by extremes of the spike trains adds to the distance without contributing information about the spike trains. Thus, the integrating interval should preferably be chosen as the interval covered by the extremes of the considered set of spike trains.

We have introduced localized versions of our metrics, which, depending on the localization kernel $\mathscr{L}$, could have a biologically-relevant interpretation of measuring the differences between two spike trains as they are perceived at a particular moment in time by a neuron receiving these spike trains.

6 Conclusion

In conclusion, we have introduced here a new class of spike train metrics, inspired by the Pompeiu-Hausdorff distance. The max-metric and the modulus-metric show a higher sensitivity than existing metrics to the precise timings of spikes in the considered spike trains. The modulus-metric does not depend on any parameters and can be computed using a fast algorithm. All these suggest the modulus-metric as a preferred metric when there is a need to measure differences between two spike trains.

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Appendix

A Equivalent Hausdorff form

Proposition A.1. *The Pompeiu-Hausdorff metric*

$$h(T, \bar{T}) = \sup_{x \in \mathbb{R}} \left| d(x, T) - d(x, \bar{T}) \right| \tag{35}$$

can be equivalently expressed as

$$h(T,\bar{T}) = \sup_{x\in[a,b]} \left|d(x,T) - d(x,\bar{T})\right|. \tag{36}$$

Proof. We need to show that restricting the interval in Eq. 35 is acceptable since the supremum is achieved on the interval $[a,b]$.

We first consider $x \in (-\infty, a]$. Because $\forall\ t \in T,\ x \le a \le t^{(1)} \le t$, we have $d(x,T) = t^{(1)} - x$ and $d(a,T) = t^{(1)} - a$. Because $\forall\ \bar{t} \in \bar{T},\ x \le a \le \bar{t}^{(1)} \le \bar{t}$, we have $d(x,\bar{T}) = \bar{t}^{(1)} - x$ and $d(a,\bar{T}) = \bar{t}^{(1)} - a$. Thus

$$\sup_{x\in(-\infty,a]} \left|d(x,T) - d(x,\bar{T})\right| = \sup_{x\in(-\infty,a]} \left|(t^{(1)} - x) - (\bar{t}^{(1)} - x)\right| = \left|t^{(1)} - \bar{t}^{(1)}\right| \tag{37}$$

$$\sup_{x\in(-\infty,a]} \left|d(x,T) - d(x,\bar{T})\right| = \left|d(a,T) - d(a,\bar{T})\right|. \tag{38}$$

Analogously, for $x \in [b,\infty)$ we have

$$\sup_{x\in[b,\infty)} \left|d(x,T) - d(x,\bar{T})\right| = \left|d(b,T) - d(b,\bar{T})\right|. \tag{39}$$

From Eqs. 38 and 39,

$$\sup_{x\in\mathbb{R}} \left|d(x,T) - d(x,\bar{T})\right| = \sup_{x\in[a,b]} \left|d(x,T) - d(x,\bar{T})\right|. \tag{40}$$

□

B Analysis of the max-metric

Proposition B.1. $d_m(T,\bar{T}) < \infty$.

Proof. From Eq. 8, for every $x \in [a,b]$ we have

$$\left|d(x,T) - d(x,\bar{T})\right| \le h(T,\bar{T}). \tag{41}$$

Multiplying by $\mathcal{H}(\cdot)$, which is positive, we obtain, $\forall s \in [a,b]$,

$$\left|d(x,T) - d(x,\bar{T})\right|\ \mathcal{H}(|s-x|) \le h(T,\bar{T})\ \mathcal{H}(|s-x|). \tag{42}$$

By taking the supremum and integrating, the equation above becomes

$$\int_a^b \sup_{x\in[a,b]} \left\{\left|d(t,T) - d(t,\bar{T})\right|\ \mathcal{H}(|s-x|)\right\} \mathrm{d}s \le \int_a^b \sup_{x\in[a,b]} \left\{h(T,\bar{T})\ \mathcal{H}(|s-x|)\right\} \mathrm{d}s. \tag{43}$$

The left side of the equation above is the max-metric d_m (Eq. 15). Because $h(T,\bar{T})$ is independent of s and x,

$$d_m(T,\bar{T}) \le h(T,\bar{T}) \int_a^b \sup_{x\in[a,b]} \mathcal{H}(|s-x|)\mathrm{d}s. \tag{44}$$

Since $h(T,\bar{T}) \le b-a$ and $\mathcal{H}(y) < m$, $\forall y \in [0,b-a]$ (Eq. 12), we obtain

$$d_m(T,\bar{T}) \le (b-a)^2\ m < \infty. \tag{45}$$

□

Proposition B.2. *$d_m : \mathscr{S}_{[a,b]} \times \mathscr{S}_{[a,b]} \to \mathbb{R}^+$ is a metric.*

Proof. In order to show that $d_m(\cdot,\cdot)$ is a metric we need to prove that it is non-negative, that $d_m(T,\bar{T}) = 0 \Leftrightarrow T = \bar{T}$ for any $T, \bar{T} \in \mathscr{S}_{[a,b]}$, that is symmetric, and that it satisfies the triangle inequality.

It is trivial to show that $d_m(T,\bar{T})$ is non-negative, symmetric, and that if $T = \bar{T} \Rightarrow d_m(T,\bar{T}) = 0$.

In order to prove that $d_m(T,\bar{T}) = 0 \Rightarrow T = \bar{T}$ we use a *reductio ad absurdum* argument. Assume that $d_m(T,\bar{T}) = 0$ with $T \neq \bar{T}$. Then, there must be at least one spike in one of the two spike trains that is not in the other. Consider that this spike belongs to T; in the case that it belongs to $\bar{T}$, the proof is analogous. Let u be the timing of this spike, $u \in T \setminus \bar{T}$. Because $u \in [a,b]$, we have, $\forall s \in [a,b]$,

$$\sup_{x\in[a,b]} \left\{\left|d(x,T) - d(x,\bar{T})\right| \mathscr{H}(|s-x|)\right\} \geq \left|d(u,T) - d(u,\bar{T})\right| \mathscr{H}(|s-u|). \tag{46}$$

Because $u \in T$, $d(u,T) = 0$. Thus

$$\sup_{x\in[a,b]} \left\{\left|d(x,T) - d(x,\bar{T})\right| \mathscr{H}(|s-x|)\right\} \geq d(u,\bar{T})\, \mathscr{H}(|s-u|). \tag{47}$$

By integrating the equation above, we obtain

$$d_m(T,\bar{T}) \geq d(u,\bar{T}) \int_a^b \mathscr{H}(|s-u|)\, \mathrm{d}s. \tag{48}$$

Because $u \notin \bar{T}$, $d(u,\bar{T}) > 0$. Also considering Eq. 9, we have

$$d(u,\bar{T}) \int_a^b \mathscr{H}(|s-u|)\, \mathrm{d}s > 0. \tag{49}$$

Thus, from Eqs. 48 and 49, $d_m(T,\bar{T}) > 0$. Since we considered that $d_m(T,\bar{T}) = 0$, this cannot be true. Hence, $T \subseteq \bar{T}$. Likewise, one can show that $\bar{T} \subseteq T$ and so $T = \bar{T}$.

In order to prove the triangle inequality, consider $\hat{T} \in \mathscr{S}_{[a,b]}$. We have, $\forall x \in \mathbb{R}$,

$$|d(x,T) - d(x,\hat{T}) + d(x,\hat{T}) - d(x,\bar{T})| \leq |d(x,T) - d(x,\hat{T})| + |d(x,\hat{T}) - d(x,\bar{T})|. \tag{50}$$

Because $\sup_x(f(x)+g(x)) \leq \sup_x(f(x)) + \sup_x(g(x))$ for any two functions f and g, it follows that

$$\begin{aligned}\sup_{x\in[a,b]} \left\{\left|d(x,T) - d(x,\bar{T})\right| \mathscr{H}(|s-x|)\right\} \leq& \sup_{x\in[a,b]} \left\{\left|d(x,T) - d(x,\hat{T})\right| \mathscr{H}(|s-x|)\right\} + \\ &\sup_{x\in[a,b]} \left\{\left|d(x,\hat{T}) - d(x,\bar{T})\right| \mathscr{H}(|s-x|)\right\}.\end{aligned} \tag{51}$$

After integration, it results that

$$d_m(T,\bar{T}) \leq d_m(T,\hat{T}) + d_m(\hat{T},\bar{T}). \tag{52}$$

With this final equality we have shown that the distance is indeed a metric and the proof ends. □

Proposition B.3. *The metric $d_m : \mathscr{S}_{[a,b]} \times \mathscr{S}_{[a,b]} \to \mathbb{R}$ is topologically equivalent to the Pompeiu-Hausdorff distance.*

Proof. In order to show that the metrics d_m and h are topologically equivalent it is sufficient to prove that the identity function $i_{\mathscr{S}_{[a,b]}} : (\mathscr{S}_{[a,b]}, d_m) \to (\mathscr{S}_{[a,b]}, h)$, $i_{\mathscr{S}_{[a,b]}}(T) = T$ and its inverse are both continuous (O'Searcoid, 2007, p. 229; Deza and Deza, 2009, p. 12).

We first show that $i_{\mathscr{S}_{[a,b]}}$ is continuous, which is equivalent to: $\forall\ T \in \mathscr{S}_{[a,b]}$, $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ such that $\forall\ \bar{T} \in \mathscr{S}_{[a,b]}$ with $d_m(T,\bar{T}) \le \delta$ we have $h(T,\bar{T}) \le \epsilon$. We choose $\delta(\epsilon) = \epsilon\, A(T,\bar{T})$ with

$$A(T,\bar{T}) = \inf_{t \in T \cup \bar{T}} \int_a^b \mathscr{H}(|s-t|)\, \mathrm{d}s. \tag{53}$$

From Eq. 9, we have that $A(T,\bar{T}) > 0$. From Eq. 15, for all $t \in T$,

$$d_m(T,\bar{T}) \ge \int_a^b |d(t,T) - d(t,\bar{T})|\, \mathscr{H}(|s-t|)\mathrm{d}s \tag{54}$$

$$= \int_a^b d(t,\bar{T})\, \mathscr{H}(|s-t|)\mathrm{d}s \tag{55}$$

$$\ge d(t,\bar{T}) \inf_{t' \in T} \int_a^b \mathscr{H}(|s-t'|)\, \mathrm{d}s \tag{56}$$

and thus

$$d_m(T,\bar{T}) \ge \sup_{t \in T} d(t,\bar{T}) \inf_{t' \in T} \int_a^b \mathscr{H}(|s-t'|)\, \mathrm{d}s \tag{57}$$

$$d_m(T,\bar{T}) \ge \sup_{t \in T} d(t,\bar{T}) \inf_{t' \in T \cup \bar{T}} \int_a^b \mathscr{H}(|s-t'|)\, \mathrm{d}s. \tag{58}$$

Analogously, for all $\bar{t} \in \bar{T}$,

$$d_m(T,\bar{T}) \ge \sup_{\bar{t} \in \bar{T}} d(\bar{t},T) \inf_{t' \in T \cup \bar{T}} \int_a^b \mathscr{H}(|s-t'|)\, \mathrm{d}s. \tag{59}$$

Taking the max value in Eqs. 58 and 59 we obtain

$$d_m(T,\bar{T}) \ge \max\left\{\sup_{t \in T} d(t,\bar{T}),\ \sup_{\bar{t} \in \bar{T}} d(\bar{t},T)\right\} \inf_{t' \in T \cup \bar{T}} \int_a^b \mathscr{H}(|s-t'|)\, \mathrm{d}s. \tag{60}$$

From Eq. 6 it follows that

$$d_m(T,\bar{T}) \ge h(T,\bar{T}) \inf_{t \in T \cup \bar{T}} \int_a^b \mathscr{H}(|s-t|)\, \mathrm{d}s. \tag{61}$$

Because $\epsilon\, A(T,\bar{T}) = \delta(\epsilon) \ge d_m(T,\bar{T})$, from the last equation we have that $\epsilon\, A(T,\bar{T}) \ge h(T,\bar{T})\, A(T,\bar{T})$. Thus $h(T,\bar{T}) \le \epsilon$ since $A(T,\bar{T}) > 0$.

It remains to show that $i^{-1}_{\mathscr{S}_{[a,b]}}$ is continuous, which is equivalent to: $\forall\ T \in \mathscr{S}_{[a,b]}$, $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon) > 0$ such that $\forall\ \bar{T} \in \mathscr{S}_{[a,b]}$ with $h(T,\bar{T}) \le \delta$ we have $d_m(T,\bar{T}) \le \epsilon$. We choose $\delta(\epsilon) = \epsilon / B(a,b)$ with

$$B(a,b) = \int_a^b \sup_{x \in [a,b]} \mathscr{H}(|s-x|)\, \mathrm{d}s. \tag{62}$$

We have

$$B(a,b) \geq \sup_{x\in[a,b]} \int_a^b \mathscr{H}(|s-x|)\,\mathrm{d}s. \tag{63}$$

Considering Eq. 9, we have $B(a,b) > 0$. From Eq. 15 we have

$$d_m(T,\bar{T}) \leq \int_a^b \sup_{x\in[a,b]} \left|d(x,T) - d(x,\bar{T})\right| \sup_{x\in[a,b]} \mathscr{H}(|s-x|)\,\mathrm{d}s \tag{64}$$

$$d_m(T,\bar{T}) \leq \sup_{x\in[a,b]} \left|d(x,T) - d(x,\bar{T})\right| \int_a^b \sup_{x\in[a,b]} \mathscr{H}(|s-x|)\,\mathrm{d}s. \tag{65}$$

From Eqs. 8 and 63 we have

$$d_m(T,\bar{T}) \leq h(T,\bar{T})\,B(a,b). \tag{66}$$

Since $h(T,\bar{T}) \leq \delta(\epsilon) = \epsilon/B(a,b)$ and $B(a,b) > 0$, we get $h(T,\bar{T})\,B(a,b) \leq \epsilon$ and, finally, $d_m(T,\bar{T}) \leq \epsilon$. □

C Analysis of the convolution max-metric

Lemma C.1. *Let* $g\colon [a,b] \to \mathbb{R}$ *be a continuous function and* $h\colon [0,b-a] \to \mathbb{R}$ *be a continuous function which is derivable on* $(0,b-a)$ *and has bounded derivative. Then the function* $q\colon [a,b] \to \mathbb{R}$,

$$q(s) = \sup_{x\in[a,b]} \left[g(x)\,h(|s-x|)\right] \tag{67}$$

is continuous on $[a,b]$.

Proof. Consider $s_0 \in [a,b]$. We need to show that $\forall\,\epsilon > 0$, $\exists\,\delta(\epsilon) > 0$ such that $\forall\,s \in (s_0-\delta, s_0+\delta)\bigcap[a,b]$, we have

$$\left|\sup_{x\in[a,b]} \left[g(x)\,h(|s-x|)\right] - \sup_{x\in[a,b]} \left[g(x)\,h(|s_0-x|)\right]\right| < \epsilon. \tag{68}$$

We have

$$g(x)\,h(|s-x|) - g(x)\,h(|s_0-x|) \leq \left|g(x)\,h(|s-x|) - g(x)\,h(|s_0-x|)\right| \tag{69}$$

$$g(x)\,h(|s-x|) \leq \left|g(x)\,[h(|s-x|) - h(|s_0-x|)]\right| + g(x)\,h(|s_0-x|) \tag{70}$$

$$g(x)\,h(|s-x|) \leq |g(x)|\,|h(|s-x|) - h(|s_0-x|)| + g(x)\,h(|s_0-x|). \tag{71}$$

The function g is bounded since it is continuous on a compact interval (Protter, 1998, p. 56). We denote by M the bound of the absolute value of g, i.e. $|g(x)| \leq M$, $\forall x \in [a,b]$. We denote by L the bound of the absolute value of the derivative of h, i.e. $|h(x) - h(y)| \leq L\,|x-y|$, $\forall\,x,y \in [0,b-a]$. Let $\epsilon > 0$ and $\delta = \epsilon/(M\,L)$. Then for all $s \in (s_0-\delta, s_0+\delta)\bigcap[a,b]$,

$$|h(|s-x|) - h(|s_0-x|)| \leq L\,||s-x| - |s_0-x|| \tag{72}$$

$$|g(x)|\,|h(|s-x|) - h(|s_0-x|)| \leq M\,L\,||s-x| - |s_0-x||. \tag{73}$$

From Eqs. 71 and 73,

$$g(x)\,h(|s-x|) \leq M\,L\,||s-x| - |s_0-x|| + g(x)\,h(|s_0-x|). \tag{74}$$

Because $\forall u, v \in \mathbb{R}$ we have $||u| - |v|| \leq |u - v|$, it follows that

$$g(x)\ h(|s - x|) \leq M\ L\ |s - s_0| + g(x)\ h(|s_0 - x|). \tag{75}$$

Because $|s - s_0| < \delta = \epsilon/(M\ L)$,

$$\begin{aligned} g(x)\ h(|s - x|) &< M\ L\ \frac{\epsilon}{M\ L} + g(x)\ h(|s_0 - x|) \\ &= \epsilon + g(x)\ h(|s_0 - x|). \end{aligned} \tag{76}$$

Applying supremum to the equation above we obtain

$$\sup_{x \in [a,b]} \left[g(x)\ h(|s - x|)\right] < \epsilon + \sup_{x \in [a,b]} \left[g(x)\ h(|s_0 - x|)\right]. \tag{77}$$

It follows that

$$\sup_{x \in [a,b]} \left[g(x)\ h(|s - x|)\right] - \sup_{x \in [a,b]} \left[g(x)\ h(|s_0 - x|)\right] < \epsilon. \tag{78}$$

Analogously, by switching s and s_0 in Eq. 69 and the ensuing equations, we get

$$\sup_{x \in [a,b]} \left[g(x)\ h(|s_0 - x|)\right] - \sup_{x \in [a,b]} \left[g(x)\ h(|s - x|)\right] < \epsilon. \tag{79}$$

Thus we have proved Eq. 68 and the proof ends. □

Proposition C.1. *$d_c(T, \bar{T}) < \infty$.*

Proof. Because $\mathscr{K}$ is positive (Eq. 18) we have $f \geq 0$ and $\bar{f} \geq 0$. Moreover, $\forall x \in \mathbb{R}$,

$$\begin{aligned} \left|f(x) - \bar{f}(x)\right| &\leq \left|f(x)\right| + \left|\bar{f}(x)\right| = f(x) + \bar{f}(x) \\ \left|f(x) - \bar{f}(x)\right| &\leq \sum_{i=1}^{n} \mathscr{K}(x - t^{(i)}) + \sum_{i=1}^{\bar{n}} \mathscr{K}(x - \bar{t}^{(i)}). \end{aligned} \tag{80}$$

Since $\mathscr{K}(x) \leq p,\ \forall\ x \in \mathbb{R}$ (Eq. 18),

$$\left|f(x) - \bar{f}(x)\right| \leq p\ (n + \bar{n}). \tag{81}$$

From Eq. 22,

$$d_c(T, \bar{T}) \leq \int_a^b \sup_{x \in [a,b]} \left|f(x) - \bar{f}(x)\right| \sup_{x \in [a,b]} \mathscr{H}(|s - x|)\ \mathrm{d}s \tag{82}$$

$$d_c(T, \bar{T}) \leq p\ (n + \bar{n}) \int_a^b \sup_{x \in [a,b]} \mathscr{H}(|s - x|)\ \mathrm{d}s. \tag{83}$$

Because $\mathscr{H}(y) < m,\ \forall\ y \in [0, b - a]$ (Eq. 12), it follows that

$$d_c(T, \bar{T}) \leq p\ (n + \bar{n})\ (b - a)\ m < \infty. \tag{84}$$

□

Proposition C.2. *$d_c : \mathscr{S}_{[a,b]} \times \mathscr{S}_{[a,b]} \to \mathbb{R}^+$ is a metric.*

Proof. In order to show that $d_c(\cdot,\cdot)$ is a metric we need to prove that it is non-negative, that $d_c(T,\bar{T}) = 0 \Leftrightarrow T = \bar{T}$ for any $T, \bar{T} \in \mathscr{S}_{[a,b]}$, that it is symmetric, and that it satisfies the triangle inequality.

It is trivial to show that $d_c(T,\bar{T})$ is non-negative, symmetric, and that if $T = \bar{T} \Rightarrow d_c(T,\bar{T}) = 0$. In order to prove that $d_c(T,\bar{T}) = 0 \Rightarrow T = \bar{T}$ we use a *reductio ad absurdum* argument. Assume that $d_c(T,\bar{T}) = 0$ with $T \neq \bar{T}$. For $s \in [a,b]$ let

$$q(s) = \sup_{x\in[a,b]} \left\{ \left| f(x) - \bar{f}(x) \right| \mathscr{H}(|s-x|) \right\}. \tag{85}$$

Because $|f(x) - \bar{f}(x)|$ is continuous, from the properties of $\mathscr{H}$ and Lemma C.1 we obtain that q is continuous. Because of the properties of $\mathscr{K}$, $T \neq \bar{T} \Rightarrow \exists\, x \in [a,b]$ such that $f(x) \neq \bar{f}(x)$; and because $\mathscr{H}$ is strictly positive, it follows that q is not zero everywhere, $\exists x \in [a,b]$ such that $q(x) > 0$. Because q is continuous, it follows that

$$d_c(T,\bar{T}) = \int_a^b q(s)\, \mathrm{d}s > 0 \tag{86}$$

which contradicts the hypothesis that $d_c(T,\bar{T}) = 0$. Hence, $T = \bar{T}$.

In order to prove the triangle inequality, consider $\hat{T} \in \mathscr{S}_{[a,b]}$. We have, $\forall x, s \in [a,b]$,

$$\left| f(x) - \hat{f}(x) + \hat{f}(x) - \bar{f}(x) \right| \mathscr{H}(|s-x|) \leq \left(\left| f(x) - \hat{f}(x) \right| + \left| \hat{f}(x) - \bar{f}(x) \right| \right) \mathscr{H}(|s-x|). \tag{87}$$

Because $\sup_s(g(s) + h(s)) \leq \sup_s(g(s)) + \sup_s(h(s))$ for any two functions g and h, it follows that, $\forall s \in [a,b]$,

$$\begin{aligned} \sup_{x\in[a,b]} \left\{ \left| f(x) - \bar{f}(x) \right| \mathscr{H}(|s-x|) \right\} \leq & \sup_{x\in[a,b]} \left\{ \left| f(x) - \hat{f}(x) \right| \mathscr{H}(|s-x|) \right\} + \\ & \sup_{x\in[a,b]} \left\{ \left| \hat{f}(x) - \bar{f}(x) \right| \mathscr{H}(|s-x|) \right\}. \end{aligned} \tag{88}$$

After integration, it results that

$$d_c(T,\bar{T}) \leq d_c(T,\hat{T}) + d_c(\hat{T},\bar{T}). \tag{89}$$

With this final equality we have shown that the distance is indeed a metric and the proof ends. □

D Analysis of the localized max-metric

Proposition D.1. $d_l(T,\bar{T}) < \infty$.

Proof. For every $s \in [a,b]$ we have

$$\begin{aligned} h(T,\bar{T}) &= \sup_{x\in[a,b]} \left| d(x,T) - d(x,\bar{T}) \right| \\ &\geq \sup_{x\in[s,b]} \left| d(x,T) - d(x,\bar{T}) \right| \end{aligned} \tag{90}$$

$$h(T,\bar{T})\, \mathscr{L}(b-s) \geq \mathscr{L}(b-s) \sup_{x\in[s,b]} \left| d(x,T) - d(x,\bar{T}) \right|. \tag{91}$$

By integrating the last equation above, we obtain

$$\int_a^b \mathscr{L}(b-s) \sup_{x\in[s,b]} \left| d(x,T) - d(x,\bar{T}) \right| \mathrm{d}s \leq h(T,\bar{T}) \int_a^b \mathscr{L}(b-s) \mathrm{d}s. \tag{92}$$

Since $h(T,\bar{T}) \leq b-a$ and $\mathscr{L}(y) < m$, $\forall y \in [0, b-a]$ (Eq. 24), it follows that

$$d_l(T,\bar{T}) \leq m\,(b-a)^2 < \infty. \tag{93}$$

□

Proposition D.2. *$d_l : \mathscr{S}_{[a,b]} \times \mathscr{S}_{[a,b]} \to \mathbb{R}^+$ is a metric.*

Proof. In order to show that $d_l(\cdot,\cdot)$ is a metric we need to prove that it is non-negative, that $d_l(T,\bar{T}) = 0 \Leftrightarrow T = \bar{T}$ for any $T, \bar{T} \in \mathscr{S}_{[a,b]}$, that it is symmetric, and that it satisfies the triangle inequality.

It is trivial to show that $d_l(T,\bar{T})$ is non-negative, symmetric, and that if $T = \bar{T} \Rightarrow d_l(T,\bar{T}) = 0$.

In order to prove that $d_l(T,\bar{T}) = 0 \Rightarrow T = \bar{T}$ we use a *reductio ad absurdum* argument. Assume that $d_l(T,\bar{T}) = 0$ with $T \neq \bar{T}$. Then, there must be at least one spike in one of the two spike trains that is not in the other. Consider that this spike belongs to T; in the case it belongs to $\bar{T}$, the proof is analogous. Let u be the timing of this spike, $u \in T \setminus \bar{T}$.

First, we consider the case $u > a$. We have, $\forall s \in [a,u]$,

$$\sup_{x\in[s,b]} \left|d(x,T) - d(x,\bar{T})\right| \geq \left|d(u,T) - d(u,\bar{T})\right|. \tag{94}$$

Because $u \in T \setminus \bar{T}$, $d(u,T) = 0$ and $d(u,\bar{T}) > 0$. Thus,

$$\sup_{x\in[s,b]} \left|d(x,T) - d(x,\bar{T})\right| \geq d(u,\bar{T}). \tag{95}$$

Multiplying by $\mathscr{L}(\cdot)$ and integrating the above equation, we obtain

$$\int_a^u \mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,T) - d(x,\bar{T})\right| \,\mathrm{d}s \geq d(u,\bar{T}) \int_a^u \mathscr{L}(b-s)\,\mathrm{d}s. \tag{96}$$

Because

$$\int_u^b \mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,T) - d(x,\bar{T})\right| \,\mathrm{d}s \geq 0, \tag{97}$$

from Eq. 28 we have

$$d_l(T,\bar{T}) \geq \int_a^u \mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,T) - d(x,\bar{T})\right| \,\mathrm{d}s \tag{98}$$

$$d_l(T,\bar{T}) \geq d(u,\bar{T}) \int_a^u \mathscr{L}(b-s)\,\mathrm{d}s. \tag{99}$$

Because $\mathscr{L}$ is strictly positive on $(0, b-a]$ and continuous, we have $\int_a^u \mathscr{L}(b-s)\,\mathrm{d}s > 0$. Because we have $d(u,\bar{T}) > 0$, we get

$$d_l(T,\bar{T}) > 0. \tag{100}$$

Since we have considered $d_l(T,\bar{T}) = 0$, it follows that Eq. 100 cannot be true. Hence, $T \subseteq \bar{T}$.

Second, we consider the case $u = a$. Let v be the timing of the first spike in either T or $\bar{T}$, other than u. Since $T \neq \bar{T}$, $v > u$. Because

$$\int_{(u+v)/2}^b \mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,T) - d(x,\bar{T})\right| \,\mathrm{d}s \geq 0, \tag{101}$$

from Eq. 28 we have

$$d_l(T,\bar{T}) \geq \int_a^{(u+v)/2} \mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,T)-d(x,\bar{T})\right| \,\mathrm{d}s \tag{102}$$

$$d_l(T,\bar{T}) \geq \int_a^{(u+v)/2} \mathscr{L}(b-s) \sup_{x\in[s,(u+v)/2]} \left|d(x,T)-d(x,\bar{T})\right| \,\mathrm{d}s. \tag{103}$$

For all $x \in [u,(u+v)/2)$ we have $d(x,T) = x-u$, $d(x,\bar{T}) > v-x$, and because on this interval $v-x > x-u$, we have $\left|d(x,T)-d(x,\bar{T})\right| > 0$. Because $\mathscr{L}$ is strictly positive on $(0,b-a]$, we get

$$\int_a^{(u+v)/2} \mathscr{L}(b-s) \sup_{x\in[s,(u+v)/2]} \left|d(x,T)-d(x,\bar{T})\right| \,\mathrm{d}s > 0 \tag{104}$$

and thus

$$d_l(T,\bar{T}) > 0. \tag{105}$$

Since we have considered $d_l(T,\bar{T}) = 0$, it follows that Eq. 105 cannot be true. Hence, $T \subseteq \bar{T}$.

Thus, we have shown that in both the case $u > a$ and the case $u = a$ we have $T \subseteq \bar{T}$. Likewise, one can show that $\bar{T} \subseteq T$ and so $T = \bar{T}$ if $d_l(T,\bar{T}) = 0$.

In order to prove the triangle inequality consider $\hat{T} \in \mathscr{S}_{[a,b]}$. We have, $\forall x \in [a,b]$,

$$|d(x,T)-d(x,\hat{T})+d(x,\hat{T})-d(x,\bar{T})| \leq |d(x,T)-d(x,\hat{T})|+|d(x,\hat{T})-d(x,\bar{T})|. \tag{106}$$

Because $\sup_x(g(x)+h(x)) \leq \sup_x(g(x))+\sup_x(h(x))$ for any two functions g and h, it follows that, $\forall s \in [a,b]$,

$$\mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,T)-d(x,\bar{T})\right| \leq \mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,T)-d(x,\hat{T})\right| + \mathscr{L}(b-s) \sup_{x\in[s,b]} \left|d(x,\hat{T})-d(x,\bar{T})\right|. \tag{107}$$

After integration, it results that $d_l(T,\bar{T}) \leq d_l(T,\hat{T}) + d_l(\hat{T},\bar{T})$.

With this final equality we have shown that the distance is indeed a metric and the proof ends. □

E Analysis of the localized modulus-metric

Proposition E.1. $d_n(T,\bar{T}) < \infty$.

Proof. From Eq. 8, we have, $\forall s \in [a,b]$,

$$h(T,\bar{T}) \geq \left|d(s,T)-d(s,\bar{T})\right|. \tag{108}$$

Multiplying by $\mathscr{L}(\cdot)$, which is positive, we obtain

$$h(T,\bar{T})\, \mathscr{L}(b-s) \geq \mathscr{L}(b-s) \left|d(s,T)-d(s,\bar{T})\right|. \tag{109}$$

By integrating the above equation, we obtain

$$h(T,\bar{T}) \int_a^b \mathscr{L}(b-s) \,\mathrm{d}s \geq \int_a^b \mathscr{L}(b-s) \left|d(s,T)-d(s,\bar{T})\right| \,\mathrm{d}s. \tag{110}$$

Since $h(T,\bar{T}) \leq b-a$ and $\mathscr{L}(x) < m$, $\forall x \in [0, b-a]$ it follows that

$$d_n(T,\bar{T}) < m\,(b-a)^2 < \infty. \tag{111}$$

$\square$

Proposition E.2. *$d_n : \mathscr{S}_{[a,b]} \times \mathscr{S}_{[a,b]} \to \mathbb{R}^+$ is a metric.*

Proof. In order to show that $d_n(\cdot,\cdot)$ is a metric we need to prove that it is non-negative, that $d_n(T,\bar{T}) = 0 \Leftrightarrow T = \bar{T}$ for any $T,\bar{T}$, that it is symmetric, and that it satisfies the triangle inequality.

Let $T,\bar{T} \in \mathscr{S}_{[a,b]}$. It is trivial to show that $d_n(T,\bar{T})$ is non-negative and symmetric, and that $T = \bar{T} \Rightarrow d_l(T,\bar{T}) = 0$. In order to prove the converse we use a *reductio ad absurdum* argument. Assume $T \neq \bar{T}$ with $d_n(T,\bar{T}) = 0$. For $s \in [a,b]$ let

$$q(s) = \mathscr{L}(b-s)\,\left|d(s,T) - d(s,\bar{T})\right|. \tag{112}$$

Because $\left|d(s,T) - d(s,\bar{T})\right|$ is continuous and $\mathscr{L}$ is continuous it results that q is continuous. Because $T \neq \bar{T} \Rightarrow \exists\, s \in [a,b]$ such that $d(s,T) \neq d(s,\bar{T})$; because $\mathscr{L}$ is strictly positive on $(0, b-a]$, it follows that q is not zero everywhere, $\exists s \in [a,b)$ such that $q(s) > 0$. Because q is continuous, it follows that

$$d_n(T,\bar{T}) = \int_a^b q(s)\,\mathrm{d}s > 0 \tag{113}$$

which contradicts the hypothesis that $d_n(T,\bar{T}) = 0$. Hence, $T = \bar{T}$.

In order to prove the triangle inequality consider $\hat{T} \in \mathscr{S}_{[a,b]}$. We have, $\forall s \in \mathbb{R}$,

$$|d(s,T) - d(s,\hat{T}) + d(s,\hat{T}) - d(s,\bar{T})| \leq |d(s,T) - d(s,\hat{T})| + |d(s,\hat{T}) - d(s,\bar{T})|. \tag{114}$$

Multiplying by $\mathscr{L}(\cdot)$ and integrating the above equation, we obtain

$$\begin{aligned}
&\int_a^b \left|d(s,T) - d(s,\bar{T})\right|\,\mathscr{L}(b-s)\,\mathrm{d}s \leq \\
&\int_a^b \left|d(s,T) - d(s,\hat{T})\right|\,\mathscr{L}(b-s)\,\mathrm{d}s + \int_a^b \left|d(s,\hat{T}) - d(s,\bar{T})\right|\,\mathscr{L}(b-s)\,\mathrm{d}s
\end{aligned} \tag{115}$$

It results that

$$d_n(T,\bar{T}) \leq d_n(T,\hat{T}) + d_n(\hat{T},\bar{T}). \tag{116}$$

With this final inequality we have shown that the distance is indeed a metric and the proof ends. $\square$

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